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The matrix model for the barotropic equation, connections to variational discretizations, and generalizations to the shallow water equations

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The matrix model for the barotropic vorticity equation on the torus and the 2-sphere, introduced by Zeitlin, remains a reference discretization, since it provides N conserved quantities with N degrees of freedom. Modin and Vivani recently also demonstrated its relevance for the numerical study of geophysical fluid dynamics. The origins of the discretization and its connection to the Moyal bracket of quantum mechanics are, however, somewhat mysterious, hampering the prospect of generalizing the ansatz to the shallow water and primitive equations. We show how the matrix model can be understood in the framework of variational, structure preserving discretizations of fluids introduced by Pavlov and co-workers, which has recently been extended to the finite element setting by Natale and Cotter as well as Gay-Balmaz and Gawlik. Pavlov et al.'s approach is to discretely mirror the continuous theory, where the dynamics take place in the space of (divergence free) vector fields, i.e. the Lie algebra of the (volume preserving) diffeomorphism group, and the reduced Euler-Poincaré variational principle yields the dynamical equations. Specifically, one considers the representation of the group and its Lie algebra on a finite dimensional function space, i.e. through their action on scalar functions, yielding an appropriate matrix group and Lie algebra as discrete configuration space. Because of the finite dimensional setting, one has to deviate at this point from the continuous theory and introduce a non-holonomic constraint, which amounts to restricting the finite dimensional Lie algebra to elements that correspond to vector fields. The Euler-Poincaré-d'Alembert principle has consequently also to be used to obtain semi-discrete time evolution equations. A modification of this methodology is to insist on the Euler-Poincaré theory from the continuous side and modify how the Lie algebra is discretized so that it remains applicable. Specifically, one can start with the action of a symmetry group on the configuration space, e.g. SO(3) on the 2-sphere, and consider the associated infinitesimal action of the Lie algebra on functions, which corresponds to vector fields, as in the approach by Pavlov et al. When the action admits a momentum map, it can equivalently be written using the Poisson bracket and Hamiltonians linear in the Lie algebra. Building on this and requiring that a generalization of the action on functions beyond linear Hamiltonians should be consistent with the group action, one is led to the iterated action of the Poisson algebra, which is equivalent to the Moyal bracket Lie algebra for the symmetry group (through the universal enveloping algebra of the original Lie algebra). When one then fixes a finite-dimensional spectral basis to discretize functions, this corresponds to a sub-algebra of gl(n).

Finally, using Euler-Poincaré theory, as in the continuous case, on this Lie sub-algebra, one obtains the matrix model by Zeitlin that retains N conserved quantities for N degrees of freedom. We hope that our rationalization of the derivation of the matrix model opens up the possibility to generalize it to other equations for geophysical fluid dynamics, and we discuss possible directions for the shallow water and primitive equations.